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# Extremum Principles for Some Nonlinear Heat Transfer Problems

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SUMMARY
Recent results on extremum principles for various nonlinear boundary value problems are applied to heat transfer problems involving space radiators such as fins and other parts of spacecraft. The results are illustrated by obtaining quite accurate variational solutions for such problems involving the fourth-power law of radiation.

## 1. Introduction

In a recent series of publications [1, 2, 3, 4], dual extremum principles have been developed for a wide class of linear and nonlinear boundary value problems, including a variety of boundary conditions. In many cases these principles provide upper *and* lower bounds to a basic functional or potential, and the theory is therefore particularly valuable when the basic potential represents a physical quantity of interest such as energy, capacity, or absorption probability [cf. 2]. In some problems, especially nonlinear ones, attention is centred on obtaining an approximate solution to the boundary value problem itself, and dual extremum principles, if they can be found, provide the basis of variational methods which can lead to an accurate variational solution.

The purpose of this paper is to present dual extremum principles which are relevant to nonlinear problems in heat transfer. The results are illustrated by two applications, one concerning a nonlinear fin problem with radiating tip condition, and the other concerning the temperature distribution on the surface of a thin-walled spherical spacecraft. Each of these problems involves the fourth-power law of radiation.

To make the paper reasonably self-contained we begin with a brief summary of the main results in the theory of dual extremum principles [2, 4].

## 2. A Class of Problems

It is convenient to consider boundary value problems with equations of the form [1, 2, 4]

$$T^* T\varphi = f(\varphi) \text{ in } V \tag{2.1}$$

with boundary conditions

$$\sigma_T \varphi = \sigma_T \varphi_B \text{ on } S_1 \tag{2.2}$$

$$\sigma_T^* T\varphi = \sigma_T^* u_B \text{ on } S_2 \tag{2.3}$$

$$\sigma_T^* T\varphi = b(\varphi) \text{ on } S_3 \tag{2.4}$$

Here $V$ is some region of space and $S_1 + S_2 + S_3 = S$ makes up the boundary of $V$. The linear operator $T$ has an adjoint $T^*$ defined by the relation [cf. 2]

$$\int_V u T\varphi \, dV = \int_V (T^* u)\varphi \, dV + \int_S u \sigma_T \varphi \, dS . \tag{2.5}$$

The linear operator $\sigma_T$ restricted to $S$ has an adjoint $\sigma_T^*$ defined by

$$\int_S u\sigma_T \varphi \, dS = \int_S (\sigma_T^* u)\varphi \, dS \,. \tag{2.6}$$

In equations (2.1)–(2.4) the functions $\varphi_B$ and $u_B$ are prescribed functions and $f(\varphi)$ and $b(\varphi)$ are known differentiable functions of $\varphi$. An example of operators $T$ and $T^*$ which satisfy (2.5), and which will be used later in the paper, is provided by

$$T = \frac{d}{dx}, \quad T^* = -\frac{d}{dx} \quad \text{on} \quad V = (a, b) \,, \tag{2.7}$$

$$\sigma_T = \begin{cases} +1 & x = b \\ -1 & x = a \end{cases} \tag{2.8}$$

where

$$\int_S u\sigma_T \varphi \, dS = (u\varphi)_{x=b} - (u\varphi)_{x=a} \,. \tag{2.9}$$

To introduce associated variational principles we rewrite equations (2.1)–(2.4) in canonical form by taking

$$T\varphi = u = \frac{\partial H}{\partial u} \qquad \text{in} \ V \,, \tag{2.10}$$

$$T^* u = f(\varphi) = \frac{\partial H}{\partial \varphi} \quad \text{in} \ V \,, \tag{2.11}$$

with

$$\sigma_T \varphi = \sigma_T \varphi_B \ \text{ on } \ S_1 \,, \tag{2.12}$$

$$\sigma_T^* u = \sigma_T^* u_B \ \text{ on } \ S_2 \,, \tag{2.13}$$

$$\sigma_T^* u = b(\varphi) \ \text{ on } \ S_3 \,. \tag{2.14}$$

A suitable Hamiltonian $H$ in equations (2.10) and (2.11) is given by

$$H(u, \varphi) = \tfrac{1}{2}u^2 + F(\varphi) \,, \tag{2.15}$$

where

$$F(\varphi) = \int^{\varphi} f(\theta) d\theta \,. \tag{2.16}$$

## 3. Variational Principles

The variational significance of systems of equations of the form (2.10) and (2.11) was first observed by Noble (see [2] for discussion). Following the work of Arthurs [1, 2, 3, 4] on various boundary conditions we introduce an action functional $I(U, \Phi)$ of the form

$$\begin{aligned} I(U, \Phi) = {} & \int_T [UT\Phi - H(U, \Phi)] dV - \int_{S_1} U\sigma_T(\Phi - \varphi_B) dS \\ & - \int_{S_2} (\sigma_T^* u_B)\Phi \, dS - \int_{S_3} B(\Phi) dS \end{aligned} \tag{3.1}$$

$$\begin{aligned} = {} & \int_V [(T^* U)\Phi - H(U, \Phi)] dV + \int_{S_1} (\sigma_T^* U)\varphi_B \, dS \\ & + \int_{S_2} [\sigma_T^*(U - u_B)] \Phi \, dS + \int_{S_3} [(\sigma_T^* U)\Phi - B(\Phi)] dS \end{aligned} \tag{3.2}$$

by equation (2.5), where

$$B(\varphi) = \int^{\varphi} b(\theta) d\theta \,, \tag{3.3}$$

and $H(U, \Phi)$ is defined by equation (2.15).

It then follows that

3(*a*). *Variational principle*: For arbitrary independent functions $U$, $\Phi$ the functional $I(U, \Phi)$ is stationary at $(u, \varphi)$, the solution pair of the boundary value problem described by equations (2.10)–(2.14).

Next, we consider extremum principles, that is maximum and minimum principles, associated with our class of boundary value problems.

3(*b*). *First extremum principle*: Using equation (3.1) we define a functional $J(\Phi)$ as follows:

$$J(\Phi)=I(U(\Phi), \Phi)\,, \qquad U(\Phi)=T\Phi \tag{3.4}$$

where $\Phi$ is any admissible function which satisfies the condition

$$\sigma_T\Phi=\sigma_T\varphi_B \quad \text{on} \quad S_1\,. \tag{3.5}$$

Then we see that

$$J(\Phi) = \int_V [\tfrac{1}{2}(T\Phi)^2-F(\Phi)]\,dV - \int_{S_2}(\sigma_T^* u_B)\Phi\, dS - \int_{S_3} B(\Phi)\,dS \tag{3.6}$$

$$= I(u, \varphi)+\Delta J\,, \tag{3.7}$$

where

$$\Delta J=\tfrac{1}{2}\int_V \left\{(T(\Phi-\varphi))^2-(\Phi-\varphi)^2\,\frac{\overline{df}}{d\varphi}\right\}dV-\tfrac{1}{2}\int_{S_3}(\Phi-\varphi)^2\,\frac{\overline{db}}{d\varphi}\,dS\,, \tag{3.8}$$

the bar over a derivative indicating that it is to be evaluated for some function $\varphi+\eta(\Phi-\varphi)$, $0<\eta<1$. Equation (3.7) shows that $J(\Phi)$ is stationary at $\varphi$. Further, if

$$\frac{df(\theta)}{d\theta} \leqq 0 \text{ in } V \text{ for all } \theta\,, \tag{3.9}$$

and

$$\frac{db(\theta)}{d\theta} \leqq 0 \text{ on } S_3 \text{ for all } \theta\,, \tag{3.10}$$

we see from equation (3.8) that

$$\Delta J \geqq 0 \text{ for all } \Phi \tag{3.11}$$

and by (3.7) we have the global minimum principle

$$I(u, \varphi) \leqq J(\Phi) \tag{3.12}$$

for all admissible functions $\Phi$ satisfying condition (3.5). This result (3.12) is a global version [4] of the minimum principle given in [1, 2].

3(*c*). *Second extremum principle*: Using equation (3.2) we define a functional $G(U)$ as follows:

$$G(U)=I(U, \Phi(U))\,, \quad \Phi(U)=\begin{cases} f^{-1}(T^*U) & \text{in } V \\ b^{-1}(\sigma_T^* U) & \text{on } S_3 \end{cases} \tag{3.13}$$

where $U$ is any admissible function satisfying the condition

$$\sigma_T^* U=\sigma_T^* u_B \text{ on } S_2\,. \tag{3.14}$$

Then we see that

$$G(U) = \int_V \{(T^*U)f^{-1}(T^*U)-\tfrac{1}{2}U^2-F[f^{-1}(T^*U)]\}\,dV + \int_{S_1}(\sigma_T^* U)\varphi_B\, dS$$

$$+ \int_{S_3}\{(\sigma_T^* U)b^{-1}(\sigma_T^* U)-B[b^{-1}(\sigma_T^* U)]\}\,dS \tag{3.15}$$

$$= I(u, \varphi)+\Delta G\,, \tag{3.16}$$

where

$$\Delta G = -\tfrac{1}{2}\int_T \left\{(U-u)^2-(f^{-1}(T^*U)-\varphi)^2\,\overline{\frac{df}{d\varphi}}\right\}dV-\tfrac{1}{2}\int_{S_3}[b^{-1}(\sigma_T^*U)-\varphi]^2\,\overline{\frac{db}{d\varphi}}\,dS \quad (3.17)$$

the bar over the derivatives again indicating that they are to be evaluated at some function $u+\eta(U-u)$, $0<\eta<1$. Equation (3.16) shows that $G(U)$ is stationary at $u$. In addition, if the conditions

$$\frac{df(\theta)}{d\theta}\leqq 0 \quad \text{in} \quad V \text{ for all } \theta\,, \quad (3.9)$$

and

$$\frac{db(\theta)}{d\theta}\leqq 0 \quad \text{on} \quad S_3 \text{ for all } \theta\,, \quad (3.10)$$

are satisfied we see from equation (3.17) that

$$\Delta G\leqq 0 \text{ for all } U\,, \quad (3.18)$$

and so by equation (3.16) we have the global maximum principle

$$G(U)\leqq I(u,\varphi) \quad (3.19)$$

for all admissible functions $U$ satisfying condition (3.14). This result (3.19) is a global version [4] of the maximum principle given in [1, 2].

Since by (2.10) the exact $u$ and $\varphi$ are related by $u=T\varphi$, it is useful in practice to consider $U$ to have the form $U=T\Psi$, where $\Psi$ is an approximation to $\varphi$. Then from equation (3.15) we have

$$G(T\Psi) = \int_V \{(T^*T\Psi)f^{-1}(T^*T\Psi)-\tfrac{1}{2}(T\Psi)^2-F[f^{-1}(T^*T\Psi)]\}dV$$
$$+\int_{S_1}(\sigma_T^*T\Psi)\varphi_B dS+\int_{S_3}\{(\sigma_T^*T\Psi)b^{-1}(\sigma_T^*T\Psi)-B[b^{-1}(\sigma_T^*T\Psi)]\}dS \quad (3.20)$$

and by (3.19) we have

$$G(T\Psi)\leqq I(u,\varphi)\,, \quad (3.21)$$

for all $\Psi$ satisfying, by (3.14), the condition

$$\sigma_T^*T\Psi=\sigma_T^*u_B \text{ on } S_2\,. \quad (3.22)$$

3(*d*). *Dual extremum principles.* From the results in 3(*b*) and 3(*c*) we have the global dual extremum principles [4]

$$G(T\Psi)\leqq I(u,\varphi)\leqq J(\Phi)\cdot \quad (3.23)$$

in the case when

$$\frac{df}{d\theta}\leqq 0 \quad \text{for all } \theta\,, \quad (3.24)$$

$$\frac{db}{d\theta}\leqq 0 \quad \text{for all } \theta\,, \quad (3.25)$$

for all admissible functions $\Phi$ and $\Psi$ such that

$$\sigma_T\Phi=\sigma_T\varphi_B \text{ on } S_1\,, \quad (3.26)$$

and

$$\sigma_T^*T\Psi=\sigma_T^*u_B \text{ on } S_2\,. \quad (3.27)$$

Equality holds in (3.23) when $\Phi$ and $\Psi$ are both equal to the exact function $\varphi$. This completes our summary of the relevant variational theory.

## 4. Problem I

We now turn to applications of the foregoing variational results, and our first application is concerned with a nonlinear fin problem with constant base temperature and a radiating tip condition. A regular perturbation solution of this problem was obtained recently by Bilenas and Jiji [5]. This solution took into account the interaction of conduction with radiation and its accuracy was measured by comparison with a numerical solution. For certain values of the basic parameters the perturbation solution was rather poor [cf. 5, Fig. 1], and it is therefore of some interest to consider a variational approach to the problem.

The problem is described by the nonlinear equations [5]

$$\frac{d^2\varphi}{dx^2} = \varepsilon\varphi^4+\lambda^2\varphi-\varepsilon\varphi_e^4-\lambda^2\varphi_e\,, \qquad 0\leqq x\leqq 1\,, \tag{4.1}$$

$$\varphi(0) = 1\,, \tag{4.2}$$

$$\frac{d\varphi(1)}{dx} = -\varepsilon\nu[\varphi^4(1)-\varphi_e^4]\,, \tag{4.3}$$

where $\varphi$ is the absolute temperature, $\varepsilon$ is the radiation-conduction parameter, $\lambda^2$ is the Biot modulus, $\nu$ is a fin geometry parameter, and $\varphi_e$ is the temperature of the environment.

## 5. Extremum Principles for Problem I

Problem I corresponds to the following choices in the general theory:

$$V=(0,1),\quad S_1=\{x=0\},\quad S_2=0,\quad S_3=\{x=1\}\,, \tag{5.1}$$

$$T=\frac{d}{dx},\quad T^*=-\frac{d}{dx},\quad \sigma_T=\begin{cases}+1 & \text{at } x=1\\ -1 & \phantom{\text{at }} x=0\end{cases} \tag{5.2}$$

$$f(\varphi) = -\varepsilon\varphi^4-\lambda^2\varphi+\varepsilon\varphi_e^4+\lambda^2\varphi_e\,, \tag{5.3}$$

$$\varphi_B = 1\,, \tag{5.4}$$

$$b(\varphi) = -\varepsilon\nu[\varphi^4-\varphi_e^4]\,. \tag{5.5}$$

From (5.3) and (5.5) we see that

$$\frac{df(\theta)}{d\theta} = -4\varepsilon\theta^3-\lambda^2\,, \tag{5.6}$$

$$\frac{db(\theta)}{d\theta} = -4\varepsilon\nu\theta^3\,, \tag{5.7}$$

and these are certainly nonpositive for all $\theta\geqq 0$, since $\varepsilon$, $\nu$ and $\lambda^2$ are nonnegative parameters. We can therefore use the results of section 3(*d*) and obtain global dual extremum principles. The functional $J$ is, from equation (3.6), given by

$$J(\Phi) = \int_0^1 \{[\tfrac{1}{2}(\Phi')^2-F(\Phi)]dx-[B(\Phi)]_{x=1}\,, \tag{5.8}$$

where

$$F(\Phi) = \int^{\Phi} f(\theta)d\theta = -\frac{\varepsilon}{5}\Phi^5-\frac{\lambda^2}{2}\Phi^2+(\varepsilon\varphi_e^4+\lambda^2\varphi_e)\Phi\,, \tag{5.9}$$

$$B(\Phi) = \int^{\Phi} b(\theta)d\theta = -\varepsilon\nu\left\{\frac{\Phi^5}{5}-\varphi_e^4\Phi\right\}, \tag{5.10}$$

and

$$\Phi(0)=1\,. \tag{5.11}$$

From equation (3.20) we find the functional $G$ to be

$$G(\Psi')=\int_0^1 \{(-\Psi'')f^{-1}(-\Psi'')-\tfrac{1}{2}(\Psi')^2-F[f^{-1}(-\Psi'')]\}\,dx$$
$$-\Psi'(0)-\left\{\frac{4\varepsilon v}{5}\left(\varphi_e^4-\frac{\Psi'}{\varepsilon v}\right)^{\frac{5}{4}}\right\}_{x=1}\,. \tag{5.12}$$

Here $f^{-1}(z)$ denotes the positive root $x$ of the quartic

$$\varepsilon x^4+\lambda^2 x+(z-\varepsilon\varphi_e^4-\lambda^2\varphi_e)=0\,. \tag{5.13}$$

It follows from the theory of section 3 that the global dual extremum principles

$$G(\Psi')\leqq I(u,\varphi)\leqq J(\Phi) \tag{5.14}$$

hold, provided the trial functions $\Phi$ and $\Psi$ are nonnegative.

## 6. Calculations for Problem I

We have performed calculations with trial functions of the form

$$\Phi=1-\mu_1[1-(1-x)^2]\,, \tag{6.1}$$
$$\Psi=1-\mu_2[1-(1-x)^2]\,, \tag{6.2}$$

where $\mu_1$ and $\mu_2$ were determined by optimizing the functionals $J$ and $G$. The parameter values used were those adopted in [5], namely $v=0.0208$, $\varphi_e=0.1$, $\varepsilon=0.1$, 0.2, 0.5 and $\lambda=0.5$, 1.0. The results of the variational calculation are given in Table 1. Since the upper and lower bounds $J$ and $G$ as well as the variational parameters $\mu_1$ and $\mu_2$ are close, we expect the variational solution (6.1) to be quite accurate.

In Table 2 we give the variational solution (6.1) for various values of $x$. This can be compared

TABLE 1

*Variational parameters for problem I.*

| $\lambda$ | $\varepsilon$ | $\mu_2$ | $G$ | $\mu_1$ | $J$ |
|---|---|---|---|---|---|
| 0.5 | 0.1 | 0.135 | 0.105383 | 0.130 | 0.106038 |
| | 0.2 | 0.160 | 0.117209 | 0.155 | 0.118532 |
| | 0.5 | 0.225 | 0.147140 | 0.215 | 0.151085 |
| 1.0 | 0.1 | 0.350 | 0.304178 | 0.335 | 0.310470 |
| | 0.2 | 0.360 | 0.309310 | 0.345 | 0.317137 |
| | 0.5 | 0.395 | 0.323336 | 0.370 | 0.336017 |

TABLE 2

*Variational solution $\Phi$ of problem I.*

| $x$ | $\lambda=0.5$ | | | $\lambda=1.0$ | | |
|---|---|---|---|---|---|---|
| | $\varepsilon=0.1$ | $\varepsilon=0.2$ | $\varepsilon=0.5$ | $\varepsilon=0.1$ | $\varepsilon=0.2$ | $\varepsilon=0.5$ |
| 0.2 | 0.953 | 0.944 | 0.923 | 0.879 | 0.876 | 0.867 |
| 0.4 | 0.917 | 0.901 | 0.862 | 0.786 | 0.779 | 0.763 |
| 0.6 | 0.891 | 0.870 | 0.819 | 0.719 | 0.710 | 0.689 |
| 0.8 | 0.875 | 0.851 | 0.794 | 0.678 | 0.669 | 0.645 |
| 1.0 | 0.870 | 0.845 | 0.785 | 0.665 | 0.655 | 0.630 |

with the perturbation and numerical solutions of Bilenas and Jiji [5], and it is seen that the variational function (6.1) is much more accurate than the perturbation solution for the case $\lambda = \frac{1}{2}$ and is comparable in accuracy for the case $\lambda = 1$.

One advantage of the variational approach is that it is not restricted, as the perturbation solution is, to small values of certain parameters.

## 7. Problem II

Our second application of the variational theory concerns the problem of the temperature distribution on the surface of space vehicles. A regular perturbation solution of this problem has been obtained by Hrycak [6], taking into account the direct solar radiation absorbed internally, and the energy lost internally and externally according to the Stefan–Boltzmann law, and its accuracy was measured by comparison with a numerical solution. For certain values of the basic parameters of the problem, the perturbation solution breaks down [cf. 6, Figs. 1 and 2], and it is therefore worth considering a variational approach to the problem.

Following Hrycak [6] we take the boundary value problem described by the nonlinear equation

$$\left(\frac{d^2\varphi}{d\theta^2} + \cot\theta \frac{d\varphi}{d\theta}\right)\alpha + \delta(\theta)\cos\theta + \frac{\beta}{4} - \frac{(1+\beta)}{4}\varphi^4 = 0, \qquad 0 \leqq \theta \leqq \pi, \tag{7.1}$$

with boundary conditions

$$\frac{d\varphi(0)}{d\theta} = \frac{d\varphi(\pi)}{d\theta} = 0. \tag{7.2}$$

Here $\varphi$ is the dimensionless absolute temperature, $\alpha$ is the spacecraft skin conduction parameters, $\beta$ is the relative internal radiation parameter, and

$$\delta(\theta) = \begin{cases} 0 & |\theta| \leqq \pi/2, \\ 1 & |\theta| > \pi/2. \end{cases} \tag{7.3}$$

We transform this problem by setting

$$x = \cos\theta. \tag{7.4}$$

Then equations (7.1) and (7.2) become

$$-\frac{d}{dx}\left\{(1-x^2)\frac{d\varphi}{dx}\right\} = a(x) - k\varphi^4, \tag{7.5}$$

$$(1-x^2)^{\frac{1}{2}}\frac{d\varphi}{dx} = 0 \quad \text{at} \quad x = \pm 1, \tag{7.6}$$

where

$$a(x) = \frac{1}{\alpha}\left\{\delta(x)x + \frac{\beta}{4}\right\}, \tag{7.7}$$

$$k = (1+\beta)/4\alpha, \tag{7.8}$$

with

$$\delta(x) = \begin{cases} 1 & x \geqq 0, \\ 0 & x < 0. \end{cases} \tag{7.9}$$

## 8. Extremum Principles for Problem II

Problem II corresponds to the following choices in the general theory:

$$V = (-1, 1), \quad S_1 = 0, \quad S_2 = \{x = -1, x = +1\}, \quad S_3 = 0, \tag{8.1}$$

$$T \quad = (1-x^2)^{\frac{1}{2}} \frac{d}{dx}, \quad T^* = -\frac{d}{dx}\{(1-x^2)^{\frac{1}{2}}\}, \tag{8.2}$$

$$\sigma_T \quad = \begin{cases} +(1-x^2)^{\frac{1}{2}} & \text{at} \quad x=1 \\ -(1-x^2)^{\frac{1}{2}} & \text{at} \quad x=-1 \end{cases} \tag{8.3}$$

$$f(\varphi) = a(x) - k\varphi^4 \tag{8.5}$$

$$F(\varphi) = \int^{\varphi} f(\theta)d\theta = a(x)\varphi - \frac{k}{5}\varphi^5, \tag{8.5}$$

$$u_B \quad = 0. \tag{8.6}$$

From equation (8.4) we see that

$$\frac{df(\theta)}{d\theta} = -4k\theta^3, \tag{8.7}$$

and this is nonpositive for all $\theta \geqq 0$. (Since $\alpha > 0$ and $\beta \geqq 0$ in this problem it follows from equation (7.8) that $k > 0$ holds.) We can therefore use the results of section 3 and obtain global extremum principles. By equation (3.6) the function $J$ is given by

$$J(\Phi) = \int_{-1}^{1} [\tfrac{1}{2}(1-x^2)(\Phi')^2 - a(x)\Phi + k\Phi^5/5]dx, \tag{8.8}$$

and by equation (3.20) we find the functional $G$ to be

$$G(T\Psi) = -\tfrac{1}{2}\int_{-1}^{1} (1-x^2)(\Psi')^2 dx - \frac{4}{5k^{\frac{1}{4}}}\int_{-1}^{1} \left[a(x) + \frac{d}{dx}\left\{(1-x^2)\frac{d\Psi}{dx}\right\}\right]^{\frac{5}{4}} dx \tag{8.9}$$

with

$$(1-x^2)\Psi' = 0 \text{ at } x = \pm 1. \tag{8.10}$$

It follows from the theory of section 3 that the global dual extremum principles

$$G(T\Psi) \leqq I(u, \varphi) \leqq J(\Phi) \tag{8.11}$$

hold, for all trial functions $\Phi$ and $\Psi$ which are nonnegative.

## 9. Calculations for Problem II

We have performed calculations with trial functions of the form

$$\Phi = \lambda_1 + \lambda_2(x - x^3/3) + \lambda_3(x^2 - \tfrac{1}{2}x^4), \tag{9.1}$$

$$\Psi = \mu_1 + \mu_2(x - x^3/3) + \mu_3(x^2 - \tfrac{1}{2}x^4), \tag{9.2}$$

where the parameters $\lambda_i$, $\mu_i$ $(i = 1, 2, 3)$ were determined by optimizing the functionals $J$ and $G$. The parameter values of $\alpha$ and $\beta$, which occur in equations (7.7) and (7.8), were those adopted by Hrycak [6], namely $\alpha = 0.25, 0.5$ and $\beta = 0.0, 0.05, 1.0$. The results of the variational calculation are given in Table 3. Since in this case the upper and lower bounds $J$ and $G$ as well as the varia-

TABLE 3

*Variational parameters for problem II.*

| $\alpha$ | $\beta$ | $\mu_1$ | $\mu_2$ | $\mu_3$ | $G$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $J$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.25 | 0.0 | 0.97 | 0.38 | 0.18 | −1.860 | 0.97 | 0.38 | 0.10 | −1.827 |
| | 0.5 | 0.97 | 0.26 | 0.18 | −2.591 | 0.97 | 0.30 | 0.10 | −2.575 |
| | 1.0 | 0.97 | 0.20 | 0.16 | −3.353 | 0.97 | 0.24 | 0.12 | −3.344 |
| 0.50 | 0.0 | 0.97 | 0.28 | 0.12 | −0.907 | 0.97 | 0.30 | 0.12 | −0.887 |
| | 0.5 | 0.97 | 0.20 | 0.12 | −1.280 | 0.97 | 0.24 | 0.10 | −1.271 |
| | 1.0 | 0.97 | 0.18 | .12 | −1.666 | 0.97 | 0.20 | 0.10 | −1.660 |

tional parameters $\lambda_i$ and $\mu_i$ are close, we expect the variational solution to be reasonably accurate.

In Table 4 we given the variational solution (9.1) at the points $x = \pm 1$. These values can be compared with the perturbation and numerical solutions of Hrycak [6], and it is seen that at $x=1$ (corresponding to $\theta=0$) our variational solution is lower than the values given by Hrycak,

TABLE 4

*Variational solution $\Phi$ of equation (9.1) of problem II.*

| $\alpha$ | $\beta$ | $\Phi\,(x=1)$ | $\Phi\,(x=-1)$ |
|---|---|---|---|
| 0.25 | 0.0 | 1.27 | 0.76 |
| | 0.5 | 1.22 | 0.82 |
| | 1.0 | 1.19 | 0.87 |
| 0.50 | 0.0 | 1.23 | 0.83 |
| | 0.5 | 1.18 | 0.86 |
| | 1.0 | 1.15 | 0.88 |

while at $x=-1$ $(\theta=\pi)$ our variational solution is in good agreement with the numerical solution [6]. For $1 \leqq \theta \leqq \pi$ the perturbation solution fails entirely [6].

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